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THE HELICAL MOTIONS OF A GAS, INVARIANT UNDER UNIFORM MOTION OF THE FRAME OF REFERENCE†

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An invariant submodel of ideal gas dynamics is investigated within the framework of the PODMODELI program. This submodel is constructed in a two-dimensional subalgebra, consisting of a Galilean transport operator and the sum of the transport and rotation operators. An original group property of the submodel is found: the permitted algebra is wider than the normalizer factor of the subalgebra being considered. A group classification of the submodel is carried out. A physical interpretation of the invariant solution is given. A number of exact solutions are considered: a self-similar solution and solution with a linear stream function. The conditions for the solution to be analytic on the axis of symmetry of the motion are derived.

1. THE EQUATIONS OF THE SUBMODEL

The equations of gas dynamics are considered in cylindrical coordinates t, x, r, θ; U is the projection of the velocity onto the x axis, V is the radial projection of the velocity and W is the peripheral velocity. An invariant solution is constructed using the two-dimensional subalgebra L_2 which is specified by the basis of operators $t\partial_x + \partial_u$ (the Galilean transport operator) and $\partial_x + \partial_\theta$ (the sum of the transport and rotation operators). The subalgebra is taken from the optimal system of subalgebras of the 11-dimensional Lie algebra L_{11}, which is permitted by the gas dynamics equations with an arbitrary equation of state (see [1, Table 6, N 2.10]). Calculation of the invariants of the subalgebra gives a representation of the invariant solution

$$U = t^{-1}(x-\theta) + v(t,r), \quad V = u(t,r), \quad W = w(t,r), \quad \rho = \rho(t,r), \quad p = p(t,r) \tag{1.1}$$

Substitution of this representation of solution (1.1) into the gas dynamics equations gives the equations of the submodel

$$\begin{aligned} &u_t + uu_r + \rho^{-1}p_r = r^{-1}w^2 \\ &v_t + uv_r = t^{-1}(r^{-1}w - v) \\ &w_t + uw_r = -r^{-1}uw \\ &\rho_t + u\rho_r + \rho u_r = -\rho(t^{-1} + r^{-1}u) \\ &p_t + up_r + Au_r = -A(t^{-1} + r^{-1}u); \quad A = \rho c^2 \end{aligned} \tag{1.2}$$

where $c^2 = \partial f/\partial\rho$ is the square of the velocity of sound, $p = f(\rho, S)$ is the equation of state, u, v and w are the invariant velocities, ρ is the density and p is the pressure.

The equation for the entropy S follows from system (1.2)

$$S_t + uS_r = 0 \tag{1.3}$$

The second equation of system (1.2) is separated from the system. It can be solved separately as a linear equation after the solution of the remaining equations has been found.

The twist integral

$$rw = \chi(S) \tag{1.4}$$

where χ is an arbitrary function, follows from the third equation of system (1.2) and Eq. (1.3).

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Hence the equations of the submodel can be written as a system of three equations: entropy equation (1.3) and the two further equations

$$u_t + uu_r + \rho^{-1}(f_\rho \rho_r + f_S S_r) = r^{-3}\chi^2(S), \qquad (\rho t r)_t + (u\rho t r_r) = 0 \tag{1.5}$$

The second equation of (1.5) becomes an identity after introducing the stream function ψ: $\rho tr = \omega(\psi)\psi_r$, $u\rho tr = -\omega(\psi)\psi_t$, where ω is an arbitrary non-zero function. The entropy integral

$$S = S(\psi) \tag{1.6}$$

follows from Eq. (1.3).

The first equation of (1.5) remains. After substituting the integrals this equation becomes a second-order quasilinear hyperbolic equation for the stream function

$$\psi_r^2\psi_{tt} - 2\psi_t\psi_r\psi_{tr} + (\psi_t^2 - c^2\psi_r^2)\psi_{rr} = \psi_r^3(f_s S'\omega^{-1}tr - c^2 r^{-1} - \chi^2 r^{-3}) + \psi_r^4 c^2\omega'\omega^{-1} \tag{1.7}$$

2. GROUP CLASSIFICATION

System (1.2) contains the arbitrary function $A(\rho, p)$. The question of the extension of a permitted algebra in the case of special functions A is fundamental in group analysis [2]. The equivalence transforms are as follows:

$$r' = a_1 r, \quad u' = a_1 u, \quad w' = a_1 w, \quad \rho' = a_2\rho, \quad p' = a_2 a_1^2(p + a_3), \quad A' = a_2 a_1^2 A$$

The transformations of the invariant variables are not written out here.

The kernel of the permitted algebras is defined by the operators.

$$Z = t\partial_t + r\partial_r - v\partial_v, \quad Z_\lambda = t^{-1}\lambda(S)\partial_v$$

where $\lambda(S)$ is an arbitrary function. The normalizer factor of the subalgebra L_2 in the algebra L_{11} is univariate and is specified by the operator $Z_1 = t^{-1}\partial_v$. The appearance of the infinite Abelian ideal Z_λ in the kernel is explained by the fact that the linear equation for v is separated out from system (1.2). The appearance of the dilatation operator Z is a rare exception in the case of invariant submodels for which, as a rule, only the normalizer factor is permitted.

The result of the group classification of system (1.2) is shown in Table 1, where N is the dilatation number from [1, Table 1]. A kernel occurs in all Lie algebras. The last line corresponds to the extension for the new function A which is not in [1, Table 1]. The following operator notation, which is encountered in the table, is introduced

$$Y^*_{\varphi(p)} = \rho\varphi'(p)\partial_\rho + \varphi(p)\partial_p, \quad Y_1 = r\partial_r + u\partial_u + w\partial_w - 2\rho\partial_\rho$$

$$Y_2 = t^2\partial_t + tr\partial_r + (r - tu)\partial_u - tv\partial_v - tw\partial_w - 3t\rho\partial_\rho - 5tp\partial_p$$

$$Y_3 = \partial_t - t^{-1}(v\partial_v + \rho\partial_\rho + p\partial_p)$$

The group classification of (1.3), (1.5) must be carried out using two arbitrary elements f and χ. It differs from that in Table 1 and is rather extensive to be presented in a paper in a journal. An exhaustive list of extensions has been given earlier.†

The group classification of Eq. (1.7) with three arbitrary elements f, χ and ω has not been performed.

3. CHARACTERISTICS AND EQUATIONS OF STRONG DISCONTINUITIES

The equations of the submodel (1.5), (1.3) can be written in the symmetric hyperbolic form

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Table 1

N	A	Extending operators
2	$pf(p\rho^{-\gamma})$	$(\gamma-1)Y_1+2\gamma Y_p^*$
4	$f(p)$	Y_1
5	$pf(\rho)$	$Y_1+2Y_p^*$
6	γp	$Y_p^*,\ Y_1$
7	$5/3p$	$Y_p^*,\ Y_2$
8	$f(\rho e^{-p})$	$Y_1-2Y_1^*$
9	$f(\rho)$	Y_1^*
10	$\gamma\rho^{\gamma}$	$Y_1^*,\ (\gamma-1)Y_1+2\gamma Y_p^*$
11	ρ	$Y_p^*,\ Y_1^*$
12	1	$Y_1,\ Y_1^*$
13	0	$Y_1,\ Y^*_{\varphi(p)}$
–	p	$Y_p^*,\ Y_1,\ Y_3$

$$\rho u_t+\rho u u_r+p_r=\rho\chi^2(S)r^{-3},\quad bp_t+bup_r+u_r=-t^{-1}-ur^{-1} \tag{3.1}$$

($b=\rho^{-1}c^{-2}$, $\rho=g(p,S)$ is the equation of state and $c^{-2}=g_p$).

The system has three characteristics. The characteristic form of system (3.1), (1.3) is [3]

$$C_0:\frac{dr}{dt}=u,\quad D_0S=S_t+uS_r=0$$

$$C_{\pm}:\frac{dr}{dt}=u\pm c,\quad \rho cD_{\pm}u\pm D_{\pm}p=c\rho\chi^2r^{-3}\mp\rho c^2(t^{-1}+ur^{-1})$$

$$(D_{\pm}=\partial_t+(u\pm c)\partial_r)$$

The invariant strong discontinuity $r=r_0(t)$ satisfies the following equations [3] contact discontinuity

$$u_i=r_0',\quad i=1,2;\quad [p]=p_2-p_1=0$$

shock wave

$$[v]=[w]=0$$

$$(u_2-r_0')^2=\rho_2\rho_1^{-1}[p][\rho]^{-1},\quad (u_1-r_0')^2=\rho_1\rho_2^{-1}[p][\rho]^{-1}$$

$$\varepsilon(\rho_2,p_2)-\varepsilon(\rho_1,p_1)+\frac{1}{2}(\rho_2^{-1}-\rho_1^{-1})(p_2-p_1)=0$$

The subscripts $i=1,2$ indicate the values of quantities on the different sides of the discontinuity, and $\varepsilon=\varepsilon(\rho,p)$ is the equation of state relating the energy, the density and the pressure.

4. PHYSICAL INTERPRETATION OF THE INVARIANT SOLUTION

A physical solution in accordance with formulae (1.1) corresponds to each continuous solution of system (1.2) in the domain Ω belonging to one quarter of the plane $t\geqslant t_0, r\geqslant 0$. A cylinder $r=r_1$ in the physical space $R^3(x,r,\theta)$ corresponds to each point $(t_1,r_1)\in\Omega$. Invariant functions take fixed values at this point. It follows from formulae (1.1) that V, W, ρ and p are constants on the cylinder and that the function U is only constant on the spiral lines $x=\theta+C$, $C=\text{const}$ and cannot be continuous ever the whole of the cylinder (actually, if the constant C is changed to $C + 2\pi$, the same spiral line is obtained and the function U takes an increment of $2\pi t_1^{-1}$). Thus, the physical solution, specified on the cylinders, has a strong discontinuity of the function U which cannot be invariant. Since p and ρ are continuous at this discontinuity, this is either a contact discontinuity or a wall.

Suppose the discontinuity surface is defined by the equation $F(t, x, r, \theta) = 0$ with a normal $\mathbf{n} = \nabla F/|\nabla F|$ and a velocity of motion $D_n = -F_t|\nabla F|$. The conditions on the contact discontinuity are: $u_{ni} = D_n$, $[p] = 0$ where the subscripts $i = 1, 2$ denote the side of the discontinuity. The equations for the surface

$$F_x = 0, \qquad F_t + uF_r + r^{-1}wF_\theta = 0$$

follow from this, and the solution of these equations is written in the form

$$F \equiv \theta - \left(\int J^{-1} w(t, r)\right)\Big|_{C=I(t,r)} - \varphi(I(t, r)) = 0 \tag{4.1}$$

where $I(t, r) = C$ is the integral of the equation $dr/dt = u$, $r = J(t, C)$ is the inverse of the function I and φ is an arbitrary function.

The world lines of the particles along both sides of the contact discontinuity lie in the moving surface (4.1). Actually, the world line of the particle on one side of the discontinuity satisfies the following equations, obtained from formulae (1.1)

$$\frac{dx}{dt} = \frac{x - \theta}{t} + v(t, r), \quad \frac{dr}{dt} = u(t, r), \quad r\frac{d\theta}{dt} = w(t, r) \tag{4.2}$$

with the initial conditions

$$x|_{t=t_1} = x_0, \quad r = |_{t=t_1} = r_0, \quad \theta|_{t=t_1} = \theta_0 \tag{4.3}$$

The world line of a particle on the other side of the discontinuity satisfies equations which are analogous to (4.2), where θ must be replaced by $\theta + 2\pi$ in the first equation. The initial conditions for θ in (4.3) can be taken in the form $\theta|_{t=t_1} = \theta_0 - 2\pi$. The substitution $x - 2\pi = \bar{x}$ leads to the first world line. Thus, the second world line is obtained from the first either by shifting x by -2π or by rotation around the x axis by an angle of 2π. It follows from this that both world lines lie in the surface (4.1).

5. SELF-SIMILAR SOLUTION OF THE SUBMODEL

The submodel (1.2) allows of an expansion $Z = t\partial_t - v\partial_v$ which does not enter into the normalizer factor. The self-similar solution of a submodel, constructed using Z, cannot therefore be obtained as an invariant solution with respect to some subalgebra from the optimal system for the gas dynamic equations [1, Table 6]. We will now consider this in more detail. For the time being, we will not consider the equation for v which has been separated out. The invariants of Z give a representation of the self-similar solution

$$u = u(s), \quad w = w(s), \quad \rho = \rho(s), \quad p = p(s); \quad S = S(s); \quad s = rt^{-1} \tag{5.1}$$

Substitution of expressions (5.1) into (1.2) and (1.3) leads to a system of ordinary differential equations

$$\begin{aligned} &S' = 0 \Rightarrow S(p, \rho) = S_0 \text{ is the entropy integral } (p = f(\rho, S_0)) \\ &(u - s)u' + f_\rho \rho^{-1}\rho' = s^{-1}w^2 \\ &(u - s)w' = -s^{-1}uw \\ &(u - s)\rho' + \rho u' + \rho(1 + s^{-1}u) = 0 \end{aligned} \tag{5.2}$$

$$\rho(u - s) = Cs^2w^3, \quad C = \text{const} \tag{5.3}$$

If $u = s + Cs^2w^3\rho^{-1}$ is substituted into system (5.2), two equations are obtained for determining w and ρ

$$\frac{sw'}{w}+\frac{\rho}{Csw^3}+1=0$$

$$(\rho^2 f_\rho - C^2 s^4 w^6)s\rho' = w^2\rho(\rho^2 + 2C\rho s^3 w + C^2 s^4 w^4) \tag{5.4}$$

The function $w(s)$ can be specified. The function $\rho(s)$ is then determined from the first equation, and the second equation parametrically defines the function $f_\rho(\rho)$. Hence, in the case of an arbitrary rotation, the equation of state can be chosen such that the solution is given by finite formulae. The exact solution can be determined in this way.

For example, when $w = w_0 s^k$, $k \neq -1$, the quantities

$$\rho = (C_1^{-1}s)^{3k+1}, \quad C_1^{-1-3k} = -(k+1)Cw_0^3, \quad u = \frac{k}{k+1}s$$

$$f(\rho) = C_1^{2k} w_0^2 (5k+1)^{-1} \rho^{1+2k/(3k+1)} + \frac{kC_1^2}{3(k+1)^3}\rho^{1+2/(3k+1)} + C_2$$

$$C_2 = \text{const}$$

are determined from relations (5.3) and (5.4)

6. SOLUTIONS WITH A LINEAR STREAM FUNCTION

As in the case of the one-dimensional motions of a polytropic gas [4, p. 312], we will seek the stream function of Eq. (1.7) in the form of a homogeneous linear function of the variable r

$$\psi = \frac{r}{a(t)}, \quad f = B(S)\rho^\gamma \tag{6.1}$$

Substitutive expressions (6.1) into Eq. (1.7) we obtain the relation

$$t^{\gamma-1}a^{2\gamma-1}a'' + \psi^{-\gamma}\omega^{\gamma-1}[B_\psi + \gamma B(\omega^{-1}\omega' - \psi^{-1})] = \chi^2\psi^{-4}t^{\gamma-1}a^{2\gamma-4}$$

Separation of the variables t and ψ leads to the determination of the functions

$$\chi = C_1\psi^2, \quad \omega = \psi B^{-1/\gamma}\left[C_2\left(1-\frac{1}{\gamma}\right)\int \psi B^{-1/\gamma}d\psi + C_3\right]^{1/(\gamma-1)} \tag{6.2}$$

The differential equation

$$a'' = C_1^2 a^{-3} - C_2 t^{1-\gamma} a^{1-2\gamma}, \quad C_1, C_2, C_3 = \text{const} \tag{6.3}$$

is obtained for the function $a(t)$.

If $C_2 = 0$, the solution of Eq. (6.3) is

$$a = a_0(1 + a_0^{-4}C_1^2(t-t_0)^2)^{1/2}, \quad a_0, t_0 = \text{const}$$

The quantity $a(t)$ increases hyperbolically asymptotically approaching the straight line $a_0 a = C_1(t-t_0)$ when $t \to \infty$. Note that the function $a(t)$ determines the law of motion of a particle $r = \psi_0 a(t)$ along the streamline $\psi = \psi_0$.

Positive solutions of Eq. (6.3) when $C_2 \neq 0$ for large values of t may have an oscillatory form for certain values of the constants C_1 and C_2.

We will now consider, as an example, the integrable case when $\gamma = 3/2$. The substitution $a = t^{1/2}b(t)$ leads to a solution which is expressed in terms of the elliptic integral

$$\ln\left|\frac{t}{t_0}\right| = 2\int_{b_0}^{b}\frac{bdb}{\sqrt{b^4 + C_4 b^2 + 8C_2 b - 4C_1^2}}, \quad b_0, t_0, C_1, C_2, C_4 = \text{const} \tag{6.4}$$

In the phase plane (b, c), $c = b_s$, $s = \ln t$, Eq. (6.3)

$$\frac{dc}{db} = \frac{b^4 - 4C_2 b + 4C_1^2}{4cb^3}$$

has only two real singular points $(b_1, 0)$, $(b_2, 0)$ and, moreover,

$$b_{1,2} = \frac{1}{2} D[1 \pm (-1 - 8C_2 D^{-3})^{1/2}], \quad D^6 = 16(C_2^2 + C_1^2 D^2)$$

The constants C_1, C_2 can be chosen such that $b_1 > b_2 > 0$. Then, the first singular point is the centre and the second singular point is a saddle point. So, solution (6.4) has an oscillatory form in the neighbourhood of the point $(b_1, 0)$.

7. THE HELICAL MOTION OF A GAS

We will consider a certain invariant solution of the submodel (1.3), (1.5) using the classification proposed by the first of the authors (see the paper cited in the footnote). In the case of a polytropic gas $p = S\rho^\gamma$, the system of equations (1.3), (1.5) allows of an extension $(1 - \gamma)(2t\partial_t + r\partial_r - u\partial_u) + 2\rho\partial_\rho$. The representation of the invariant solution is

$$u = t^{-1/2} u_1(s), \quad \rho = t^{1/(1-\gamma)} \rho_1(s), \quad S = S(s), \quad s = rt^{-1/2}$$

The invariant functions satisfy the system of ordinary differential equations

$$\begin{aligned} &\left(u_1 - \frac{1}{2}s\right)S' = 0 \\ &\left(s\left(u_1 - \frac{1}{2}s\right)\rho_1\right)' + \frac{2\gamma - 3}{\gamma - 1} s\rho_1 = 0 \\ &u_1 u_1' - \frac{1}{2}(su_1)' + \gamma S \rho_1^{\gamma-2}\rho_1' + \rho_1^{\gamma-1} S' = \chi^2(S)s^{-3} \end{aligned} \tag{7.1}$$

System (7.1) is integrated for $\gamma = 3/2$. There are two types of solutions: the non-isentropic case

$$u_1 = \frac{1}{2}s, \quad \rho_1 = \frac{1}{9}S^{-2/3}\left[\int_{s_0}^{s} S(s)^{-2/3}\left(\frac{1}{4}s + s^{-3}\chi^2(S(s))\right)ds\right]^2$$

where $S = S(s)$ is an arbitrary function and s_0 is a constant, and the isentropic case

$$\begin{aligned} &S = S_0, \quad u_1 = \frac{1}{2}s + C_1(s\rho_1)^{-1}, \\ &s^2\rho_1(C_0\rho_1 - 6S_0\rho_1^{3/2} - C_1) = C_1^2 + \frac{1}{2}M_0\rho_1^2 \end{aligned} \tag{7.2}$$

where $S_0 > 0$, $M_0 > 0$ and C_0, C_1 are constants.

The non-isentropic case leads to the following physical solution (1.1)

$$\begin{aligned} &U = t^{-1}(x - \theta + h(s)) + r^2\chi(S(s))\ln t \\ &V = \frac{1}{2}rt^{-1}, \quad W = r^{-1}\chi(S(s)) \\ &\rho = \frac{1}{9}t^{-2}S^{-2/3}\left[\int_{s_0}^{s} S(s_1)^{-2/3}\left(\frac{1}{4}s_1 + s_1^{-3}\chi^2(S(s_1))\right)ds_1\right]^2 \\ &S = S(s), \quad p = S\rho^{3/2} \end{aligned}$$

where $h(s)$, $S(s)$, $\chi(S)$ are arbitrary functions.

Suppose $\chi(S(s)) = ks^2$ and k is a constant. Then, the physical velocities are expressed using the formulae

$$U = t^{-1}(x - \theta + k\ln t + h(s)), \quad V = \frac{1}{2}rt^{-1}, \quad W = krt^{-1}$$

The world line of a particle which passes through the point (x_0, r_0, θ_0) at the instant of time $t_1 = 1$ is determined from (4.2) and (4.3)

$$x = x_0t + (t-1)(h_0 - \theta_0), \quad r = r_0t^{1/2}, \quad \theta = k\ln t + \theta_0, \quad h_0 = h(r_0)$$

The projection of the world line onto the space $R^3(x, r, \theta)$ is a trajectory which lies on the parabola

$$x = (x_0 + h_0 - \theta_0)(r/r_0)^2 + \theta_0 - h_0$$

The projection of the trajectory onto the plane x = const. is the spiral

$$r = r_0 \exp\frac{\theta - \theta_0}{2k}$$

We fix x_0, θ_0 and change r_0. The vertex of the paraboloid lies on the x axis at the point $\theta_0 - h(r_0)$. On reducing r_0, the paraboloid contracts to the x axis. Trajectories passing through the half-line x_0 = const, θ_0 = const, $0 \leq r_0 < \infty$ form a helical tapering surface. This surface can be a contact discontinuity or a wall and constrains the continuous physical motion of the gas.

The isentropic case describes an outflow from a three-dimensional expanding twisted source.

In fact, the function $\rho_1(s) > 0$ is defined by the implicit formula (7.2) for $s > s_0 > 0$ when $M_0 > 0$, $C_1 \neq 0$. For $C_1 > 0$ and a sufficiently large value of the constant $C_0 > 0$, the expression

$$C_0\rho_1 - 6S_0\rho_1^{3/2} - C_1 = g(\rho_1)$$

has two zeros when $\rho_1 = \rho_{11}, \rho_{12}$, $0 < \rho_{11} < \rho_{12}$. It follows from (7.2) that $\rho_1(s)$ is a two-valued bounded monotonic function which takes values between ρ_{11} and ρ_{12}. If $C_1 < 0$, the function $g(\rho_1)$ has one zero when $\rho_1 = \rho_{12} > 0$, and it follows from relations (7.2) that $\rho_1(s)$ is a two-valued monotonic function which takes values between zero and ρ_{12}. So, for each value of the constants (apart from $C_1 > 0$ and $C_0 < 0$), there are two outflows from a cylindrical twisted source which expand as $r = s_0t^{1/2}$. In the flow domain, there is a helical contact discontinuity or a wall.

8. ANALYTIC SOLUTION WITHOUT SINGULARITIES ON THE AXIS

In the case of an analytic equation of state $\rho = g(p, S)$, an analytic solution in the neighbourhood of the axis $r = 0$ can be considered. The physical conditions for solution (1.1) to have no singularities are as follows:

$$u(t, 0) = w(t, 0) = 0$$

The solution of system (1.2) is sought in the form of a power series in r. Comparison of the powers in Eq. (1.2) gives the following representation of the solution

$$u = r\sum u_kr^k, \quad w = r\sum w_kr^k, \quad \rho = \sum \rho_kr^k$$
$$p = P + r^2\sum p_kr^k, \quad S = C_0 + r\sum S_kr^k \qquad (8.1)$$

where u_k, w_k, ρ_k, p_k, S_k and P are functions of the variable t, C_0 and the summation is carried out over integral, non-negative values of k.

When $r = 0$, quantities with the zero subscript

$$\rho_0 = g(P, C_0), \quad \text{or} \quad P = f(\rho_0, C_0)$$
$$u_0 = -\frac{1}{2}(t^{-1} + \rho_0^{-1}\rho_0'), \quad w_0 = C_1t\rho_0, \qquad (8.2)$$
$$S_0 = C_2(t\rho_0)^{1/2}, \quad p_0 = \frac{1}{2}\rho_0(w_0^2 - u_0' - u_0^2)$$

where C_1, C_2 are arbitrary constants and $P(t)$ is an arbitrary constant, are determined from system (1.2) and the equation of state.

Substituting the representation of the solution in the form of (8.1) into system (1.2) and into the equation of state and comparing the coefficients of r^k we obtain equations for determining quantities with subscript k

$$\begin{aligned}
&\rho_0 u_k' + (k+2)u_0\rho_0 u_k - 2\rho_0 w_0 w_k + (u_0' + u_0^2 - w_0^2)\rho_k + (k+2)p_k = g_{1k} \\
&w_k' + 2w_0 u_k + (k+2)u_0 w_k = g_{2k} \\
&\rho_k' + (k+2)\rho_0 u_k + ((k+2)u_0 + t^{-1})\rho_k = g_{3k} \\
&S_k' + S_0 u_k + u_0 S_k = g_{4k} \\
&p_k = g_{5k}
\end{aligned} \tag{8.3}$$

The quantities g_{ik} are expressed in terms of quantities with subscripts which are smaller then k. We determine ρ_k from the last equation of (8.3), and u_k and p_k are determined from the third and first equations of (8.3), respectively. The remaining two equations are linear inhomogeneous equations for determining w_k and S_k, the solution of which has the form

$$w_k = (t\rho_0)^{1+k/2}(F_k(t) + C_{2k+1}), \quad S_k = (t\rho_0)^{1/2}(G_k(t) + C_{2k+2})$$

where C_{2k+1}, C_{2k+2} are constants and $F_k(t)$, $G_k(t)$ are certain functions.

Thus solution (8.1) is determined, apart from an arbitrary function $P(t)$ and an infinite number of constants C_0, C_1, $C_2 \ldots$

The convergence of the series (8.1) can be proved by the method of majorants in the small. The constant arbitrariness can then be specified by two arbitrary functions $\sigma(s)$, $\nu(r)$:

$$w = r\bar{w}(t,r) + r\mu(t)\sigma(r\mu(t)), \qquad S = C_0 + \bar{S}(t,r) + r\mu(t)\nu(r) \tag{8.4}$$

where $\bar{w}$, $\bar{S}$, μ are certain fixed functions. Consequently, the asymptotic behaviour of the functions w and S as $r \to 0$, compatible with (8.4), can be specified by the functions σ and ν as boundary conditions. Moreover, it is possible to specify the pressure $P(t)$ on the axis $r = 0$. The entropy is determined, apart from a constant term and the constant C_0 is therefore not essential.

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